

Finite basis problem for identities with involution.

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October 26, 2014.

Abstract

We consider associative algebras with involution over a field of characteristic zero. We proved that any algebra with involution satisfies the same identities with involution as the Grassmann envelope of some finite dimensional $(\mathbb{Z}/4\mathbb{Z})$ -graded algebra with graded involution. As a consequence we obtain the positive solution of the Specht problem for identities with involution: any associative algebra with involution over a field of characteristic zero has a finite basis of identities with involution. These results are analogs of Kemer's theorems for ordinary identities [28]. Similar results were proved also for associative algebras graded by a finite group in [1], and for abelian case in [33].

MSC: Primary 16R50; Secondary 16W20, 16W55, 16W10, 16W50

Keywords: Associative algebras, algebras with involution, identities with involution.

Introduction

The interest to involutions on associative algebras can be partially explained by their natural interconnections with various interesting and important classes of algebras which appears in different fields of mathematics and physics (see, e.g., [29]). Particularly, associative algebras with involution is the natural background for important classes of Lie and Jordan algebras ([25], [30], [37]). The identities with involution are also intensively studied last years.

In the theory of identities one of the central problem is the Specht problem. This is the problem of existence of a finite base for any system of identities. Originally this problem was formulated by W. Specht for ordinary polynomial identities of associative algebras over a field of characteristic zero [32]. This problem was positively solved by A. Kemer [28]. The solution is based on the Kemer's classification theorems. They state that any associative algebra over a field of characteristic zero is

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equivalent in terms of identities (PI-equivalent) to the Grassmann envelope of a finite dimensional superalgebra, and any finitely generated PI-algebra is PI-equivalent to a finite dimensional algebra. The classification theorems have a proper significance. They turn out to be the key tool for study of polynomial identities several last years.

The proof of the main classification theorem of Kemer consists of two principal steps: the supertrick and the PI-representability of finitely generated PI-superalgebras. On the first step the study of polynomial identities of any associative algebra is reduced to study of identities of the Grassmann envelope of a finitely generated PI-superalgebra. The second step is to prove that a finitely generated PI-superalgebra has the same $(\mathbb{Z}/2\mathbb{Z})$ -graded identities as some finite dimensional superalgebra.

Later results similar to some of the Kemer's theorems were obtained also for various classes of algebras and identities. A review of results concerning the Specht problem can be found in [11]. One of the most recent results is a positive solution of the local Specht problem for associative algebras over an associative commutative Noetherian ring with unit [7]-[14]. Graded algebras and algebras with involution were also considered with regard to this problem. The positive solution of the Specht problem and analogs of the classification theorems were obtained for graded identities of graded associative algebras over a field of characteristic zero ([1] for a grading by a finite group, and [33] for a grading by a finite abelian group). The equivalence in terms of identities with involution was proved for finitely generated and finite dimensional PI-algebras with involution [34].

The main purpose of this paper is a positive solution of the Specht problem for identities with involution. This problem can be formulated in various forms: in terms of a finite base of identities, and in terms of the Noetherian property for ideals of the free algebra which are invariant under free algebra endomorphisms. The positive answer to this question for identities with involution is equivalent to any of the following statement. Any associative algebra with involution over a field of characteristic zero has a finite base of identities with involution (all identities with involution of a $*$ -algebra follow from a finite family of $*$ -identities). Any $*$ T-ideal of the free associative algebra with involution of infinite rank over a field of characteristic zero is finitely generated as a $*$ T-ideal. Any ascending chain of $*$ T-ideals of the free associative algebra with involution of infinite rank over a field of characteristic zero eventually stabilizes. $*$ T-ideal is a $*$ -invariant two-sided ideal of the free associative algebra with involution, closed under all free algebra endomorphisms which commute with involution. See Lemma 1.1 about the structure of a $*$ T-ideal generated by a set S .

We prove in this work that any associative algebra with involution over a field of characteristic zero satisfies the same identities with involution as the Grassmann $(\mathbb{Z}/4\mathbb{Z})$ -envelope of some finitely generated $(\mathbb{Z}/4\mathbb{Z})$ -graded PI-algebra with graded involution (Theorem 4.1). This is an analog of the supertrick in the classical case. Using the recent result of the author about PI-representability of finitely generated $(\mathbb{Z}/4\mathbb{Z})$ -graded PI-algebras with graded involution [35] we obtain a version of the main classification Kemer's theorem for identities with involution (Theorem 4.2). As

a consequence we obtain the positive solution of the Specht problem for identities with involution of associative $*$ -algebras over a field of characteristic zero (Theorem 5.1).

Throughout the paper we consider associative algebras over a field F of characteristic zero. Involution of an F -algebra A is an anti-automorphism of A of the second order. If we fix an involution $*$ of an associative F -algebra A then the pair $(A, *)$ is called an *associative algebra with involution* (or *associative $*$ -algebra*). Note that an algebra with involution can be considered as an algebra with the supplementary unary linear operation $*$ satisfying identities

$$(a \cdot b)^* = b^* \cdot a^*, \quad (a^*)^* = a$$

for all $a, b \in A$.

Observe that any $*$ -algebra can be decomposed into the sum of symmetric and skew-symmetric parts. An element $a \in A$ is called *symmetric* if $a^* = a$, and *skew-symmetric* if $a^* = -a$. So, $a + a^*$ is symmetric and $a - a^*$ skew-symmetric for any $a \in A$. Thus, we have $A = A^+ \oplus A^-$, where A^+ is the subspace formed by all symmetric elements (*symmetric part*), and A^- is the subspace of all skew-symmetric elements of A (*skew-symmetric part*). We also use the notations $a \circ b = ab + ba$, and $[a, b] = ab - ba$. It is clear that the symmetric part A^+ of a $*$ -algebra A with the operation \circ is a Jordan algebra (Hermitian Jordan algebra). The skew-symmetric part A^- with the operation $[\cdot, \cdot]$ is a Lie algebra. All classical finite-dimensional simple Lie algebras over an algebraically closed field, except $sl_n(F)$, are of this type [25].

Suppose that A, B are algebras with involution. An ideal I of A invariant with respect to involution is called *$*$ -ideal*. If $I \trianglelefteq A$ is a $*$ -ideal then A/I inherits the involution of A . A homomorphism $\varphi : A \rightarrow B$ is called *$*$ -homomorphism* (*homomorphism of algebras with involution*) if it commutes with the involution. We denote by $A_1 \times \cdots \times A_\rho$ the direct product of algebras A_1, \dots, A_ρ , and by $A_1 \oplus \cdots \oplus A_\rho \subseteq A$ the direct sum of subspaces A_i of an algebra A . If τ_i is the involution of A_i ($i = 1, \dots, \rho$) then $A_1 \times \cdots \times A_\rho$ is an algebra with the involution $*$ defined by the rules $(a_1, \dots, a_\rho)^* = (\tau_1(a_1), \dots, \tau_\rho(a_\rho))$, $a_i \in A_i$.

We study identities with involution ($*$ -identities) of associative algebras with involution. The notion of identity with involution is a formal extension of the notion of ordinary polynomial identity (see, e.g., [24], [34]). A brief introduction to the notion is given in Section 1. The definition of $*$ -identity can be found also in [34] or in [24] with some more details. We refer the reader to the textbooks [17], [18], [24], and to [27], [28] concerning basic definitions, facts and properties of ordinary polynomial identities.

We also use in the proof of the classification theorem the concept of a graded identity with involution (graded $*$ -identity). This concept was developed in [35]. The principal definitions concerning this notion is also given in Section 1. In general, the concept of a graded $*$ -identity is the union of concepts of an identity with involution and of a graded identity. The information about graded identities can be found in [23], [24] and in [1], [33].

Besides the notions of the free algebra with involution, identities with involution, the free graded algebra with involution and graded identities with involution, Section 1 also contains the necessary information about graded algebras.

Properties of multilinear $*$ -polynomials and multilinear graded $*$ -polynomials alternating or symmetrizing in some set of variables are discussed in Section 3. Such polynomials appears in the study of identities as a result of application of techniques of symmetric group representations. Basic facts and notions concerning applications of representation theory for $*$ -identities can be found in [19], [22], [20], [21], [24]. Observe that in our case the application of representation theory for $*$ -identities is similar to the case of ordinary polynomial identities due to fact that the symmetric group acts by renaming of variables on a homogeneous subset of variables (on a set of symmetric variables in respect to involution or skew-symmetric). Thus in many situations we can apply the same results and arguments as in the case of ordinary polynomial identities. The book [24] contains very detailed and complete exposition of the facts and methods related to application of symmetric group representations for theory of polynomial identities. We appeal to this book when we need facts which can be directly applied in our case or arguments which can be literally repeated. We also refer the reader to [26], [16] concerning principal definitions and facts of representation theory.

Section 2 is devoted to the definition of the Grassmann envelope of a $(\mathbb{Z}/4\mathbb{Z})$ -graded algebra. Section 4 contains the classification theorems for ideals of identities with involution (Theorems 4.1, 4.2, 4.3). They are analogs of Kemer's theorems [28] for polynomial identities of associative algebras over a field of characteristic zero. The proof of Theorem 4.1 follow the scheme of the proof of the classical Kemer's theorem about Grassmann envelopes given in [24]. We adopt this proof for the case of identities with involution. Theorem 4.2 is the corollary of Theorem 4.1 and Theorem 6.2 [35]. The Specht problem solution (Theorem 5.1) for $*$ -identities is given in Section 5. The proof of Theorem 5.1 is the involution version of the original Kemer's proof [28].

Observe that the principal tool of the proof is the Grassmann envelope. Our conception of the Grassmann envelope in this work is different of the usual one. Usually one consider the Grassmann envelope $E(A) = A_{\bar{0}} \otimes E_{\bar{0}} \oplus A_{\bar{1}} \otimes E_{\bar{1}}$ for a $(\mathbb{Z}/2\mathbb{Z})$ -graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ (superalgebra). It gives super-theory. In this case a graded involution on $E(A)$ induces the superinvolution on A . A $(\mathbb{Z}/2\mathbb{Z})$ -graded linear transformation \star of the second order of a superalgebra A is called a superinvolution if

$$(a \cdot b)^\star = (-1)^{i \cdot j} b^\star a^\star \quad \forall a \in A_{\bar{i}}, b \in A_{\bar{j}}, i, j \in \{0, 1\}.$$

And vice versa, one needs a superinvolution on A to guarantee the correspondent involution on $E(A)$.

We use a slight generalization of the traditional construction based on the natural $(\mathbb{Z}/4\mathbb{Z})$ -grading of the Grassmann algebra E . We call it the Grassmann \mathbb{Z}_4 -envelope to differ it from the traditional Grassmann envelope. This construction is compatible with the usual graded involution. We think that the Specht problem for $*$ -identities

can be solved also using the traditional approach based on superinvolutions. It is possible even that the traditional approach could be more natural. But the author assume that the new construction and its connection with graded involutions on associative algebras is rather curious and worth to study.

1 Identities with involution and graded identities with involution.

Let F be a field of characteristic zero. Consider two countable sets $Y = \{y_i | i \in \mathbb{N}\}$, $Z = \{z_i | i \in \mathbb{N}\}$ of pairwise different letters, and the free associative non-unitary algebra $F\langle Y, Z \rangle$ generated by $Y \cup Z$. We can define an involution on $F\langle Y, Z \rangle$ assuming that variables from Y are symmetric, and from Z skew-symmetric

$$\left(\sum \alpha_w a_{i_1} \cdots a_{i_n}\right)^* = \sum \alpha_w a_{i_n}^* \cdots a_{i_1}^* = \sum (-1)^{\deg_Z w} \alpha_w a_{i_n} \cdots a_{i_1}, \quad \text{where} \\ y_j^* = y_j, \quad z_j^* = -z_j, \quad w = a_{i_1} \cdots a_{i_n}, \quad a_j \in Y \cup Z, \quad \alpha_w \in F. \quad (1)$$

$F\langle Y, Z \rangle$ is the free associative algebra with involution. Its elements are called **-polynomials*. The free associative algebra $F\langle X^* \rangle$ generated by the set $X^* = \{x_i, x_i^* | i \in \mathbb{N}\}$ also has an involution defined by

$$\left(\sum \alpha_w a_{i_1} \cdots a_{i_n}\right)^* = \sum \alpha_w a_{i_n}^* \cdots a_{i_1}^*, \quad \text{where} \\ (x_j)^* = x_j^*, \quad (x_j^*)^* = x_j, \quad w = a_{i_1} \cdots a_{i_n}, \quad a_j \in X^*, \quad \alpha_w \in F.$$

The equalities

$$y_i = \frac{x_i + x_i^*}{2}, \quad z_i = \frac{x_i - x_i^*}{2}; \\ x_i = y_i + z_i, \quad x_i^* = y_i - z_i \quad (2)$$

induce the isomorphism of algebras with involution $F\langle X^* \rangle$ and $F\langle Y, Z \rangle$. We use the algebra $F\langle Y, Z \rangle$ as the free associative *-algebra.

An algebra with involution A satisfies the **-identity* (or *identity with involution*) $f = 0$ for a non-trivial *-polynomial $f = f(y_1, \dots, y_n, z_1, \dots, z_m) \in F\langle Y, Z \rangle$ whenever $f(a_1, \dots, a_n, b_1, \dots, b_m) = 0$ for all elements $a_i \in A^+$, and $b_i \in A^-$. Let $\text{Id}^*(A)$ be the ideal of all identities with involution of A . Then $\text{Id}^*(A)$ is a two-sided *-ideal of $F\langle Y, Z \rangle$ closed under all *-endomorphisms of $F\langle Y, Z \rangle$. Such ideals are called **T-ideals* (see [34]). Conversely, any *T-ideal I of $F\langle Y, Z \rangle$ is the ideal of *-identities of the algebra with involution $F\langle Y, Z \rangle / I$. We denote by $*T[S]$ the *T-ideal generated by a set $S \subseteq F\langle Y, Z \rangle$. The next statement is clear due to the definition and elementary properties of a *T-ideal.

Lemma 1.1 *Let F be a field of characteristic zero. Given a set $S \subseteq F\langle Y, Z \rangle$ a polynomial $f \in F\langle Y, Z \rangle$ belongs to the *T-ideal $*T[S]$ generated by S iff f is a finite linear combination of the form*

$$f = \sum_{(u), j} \alpha_{(u), j} v_1 g_j(\tilde{u}_{j1}, \dots, \tilde{u}_{jn_j}) v_2, \quad \alpha_{(u), j} \in F. \quad (3)$$

Where $g_j = \tilde{g}_j$ or $g_j = \tilde{g}_j^*$ for the full linearization \tilde{g}_j of a multihomogeneous component of a polynomial $g \in S$; $\tilde{u}_{jl} = u_{jl} \pm u_{jl}^*$ for a monomial $u_{jl} \in F\langle Y, Z \rangle$ ($\tilde{u}_{jl} = u_{jl} + u_{jl}^*$ if the corresponding variable x_{jl} of the polynomial g_j is symmetric in respect to involution ($x_{jl} \in Y$), and $\tilde{u}_{jl} = u_{jl} - u_{jl}^*$ if $x_{jl} \in Z$ is skew-symmetric); $v_l \in F\langle Y, Z \rangle$ are monomials, possibly empty; $(u) = (v_1, \tilde{u}_{j_1}, \dots, \tilde{u}_{j_{n_j}}, v_2)$.

Proof. It is clear that the set of all polynomials of the form (3) is a \ast T-ideal, and contains S . The characteristic of the base field is zero. Therefore, any \ast T-ideal Γ contains all multihomogeneous components of its elements and their full linearizations. Particularly, if a \ast T-ideal Γ contains S then it contains also all multihomogeneous components of any $g \in S$ and their full linearizations. Moreover, any \ast -invariant evaluation of variables of a homogeneous polynomial $\tilde{g} \in \Gamma$ can be realized by a \ast -invariant evaluation of the full linearization of \tilde{g} up to a non-zero coefficient. Since the polynomials g_i are multilinear then a linear base of all their \ast -invariant evaluations is formed by their evaluations with the symmetric and skew-symmetric parts of monomials. \square

We say that a \ast -polynomial f is a *consequence* of a set $S \subseteq F\langle Y, Z \rangle$ if $f \in \ast T[S]$. We have also that $\text{Id}^\ast(A_1 \times A_2) = \text{Id}^\ast(A_1) \cap \text{Id}^\ast(A_2)$ for the direct product $A_1 \times A_2$ of arbitrary \ast -algebras A_i .

Suppose that Γ is a \ast T-ideal. A \ast -variety defined by Γ is the family of all associative \ast -algebras such that they satisfy $f = 0$ for any $f \in \Gamma$. It is denoted by \mathfrak{V}_Γ . A \ast -algebra A generates \mathfrak{V}_Γ if $\Gamma = \text{Id}^\ast(A)$. Then we write $\mathfrak{V}_\Gamma = \mathfrak{V}(A)$. The \ast -algebra $F\langle Y, Z \rangle / \Gamma$ is the relatively free algebra of the \ast -variety \mathfrak{V}_Γ . Any \ast -variety is closed under taking \ast -subalgebras, \ast -homomorphic images, and direct products. The free \ast -algebra of rank ν $F\langle Y_\nu, Z_\nu \rangle$, and the relatively free algebra of rank ν $F\langle Y_\nu, Z_\nu \rangle / (\Gamma \cap F\langle Y_\nu, Z_\nu \rangle)$ for the \ast -variety \mathfrak{V}_Γ are also considered ($Y_\nu = \{y_i | i = 1, \dots, \nu\}$, $Z_\nu = \{z_i | i = 1, \dots, \nu\}$).

Let G be a finite abelian group. An algebra A is G -graded if $A = \bigoplus_{\theta \in G} A_\theta$ is the direct sum of its subspaces A_θ satisfying $A_\theta A_\xi \subseteq A_{\theta\xi}$ for all $\theta, \xi \in G$. An element $a \in A_\theta$ is called G -homogeneous of degree $\deg_G a = \theta$. A subspace V of A is graded if $V = \bigoplus_{\theta \in G} (V \cap A_\theta)$.

Example 1.2 The free associative algebra $\mathfrak{F} = F\langle X \rangle$ generated by $X = \{x_1, x_2, \dots\}$ has the natural $(\mathbb{Z}/n\mathbb{Z})$ -grading $\mathfrak{F}_{\bar{m}} = \text{Span}_F\{x_{i_1}x_{i_2}\cdots x_{i_s} | s \equiv m \pmod{n}, \bar{m} \in \mathbb{Z}/n\mathbb{Z}\}$.

The Grassmann algebra of countable rank $E = \langle e_i, i \in \mathbb{N} | e_i e_j = -e_j e_i, \forall i, j \rangle$ has the homogeneous relations. Thus it inherits the $(\mathbb{Z}/n\mathbb{Z})$ -grading of the free algebra $E_{\bar{m}} = \text{Span}_F\{e_{i_1}e_{i_2}\cdots e_{i_s} | s \equiv m \pmod{n}, i_1 < \dots < i_s\}$. This grading is called natural.

Consider a G -graded algebra A with involution. We assume that the involution is a graded anti-automorphism of A , i.e. $A_\theta^\ast = A_\theta$ for any $\theta \in G$. This is equivalent to condition (see for instance [6]) that the subspaces A^+ , A^- are graded. Particularly, we have that $A = \bigoplus_{\theta \in G} (A_\theta^+ \oplus A_\theta^-)$, where $A^\delta = \bigoplus_{\theta \in G} A_\theta^\delta$, ($\delta \in \{+, -\}$); and

$A_\theta = A_\theta^+ \oplus A_\theta^-$, ($\theta \in G$). We say that an element $a \in A_\theta^\delta$ ($\delta \in \{+, -\}$, $\theta \in G$) is homogeneous of complete degree $\deg_{\widehat{G}} a = (\delta, \theta)$ or simply \widehat{G} -homogeneous.

Example 1.3 Consider the natural $(\mathbb{Z}/4\mathbb{Z})$ -grading on the Grassmann algebra of countable rank $E = \bigoplus_{\bar{m} \in \mathbb{Z}/4\mathbb{Z}} E_{\bar{m}}$ described in Example 1.2. Define on E the involution $*_E$ by the equalities $(e_i)^{*_E} = e_i$ for all $i \in \mathbb{N}$. This involution is called canonical. It is clear that this involution is graded. Moreover, $E^+ = E_0 \oplus E_1$, and $E^- = E_2 \oplus E_3$.

A homomorphism $\varphi : A \rightarrow B$ of two G -graded $*$ -algebras A, B is called *graded $*$ -homomorphism* if φ is graded ($\varphi(A_\theta) \subseteq A_\theta$ for any $\theta \in G$), and commutes with the involution. An ideal (a subalgebra) $I \trianglelefteq A$ of a graded algebra with involution A is *graded $*$ -ideal* (*graded $*$ -subalgebra*) if it is graded and invariant under the involution. For graded algebras with involution we consider only graded $*$ -ideals, and graded $*$ -homomorphisms. In this case the quotient algebra A/I is also a graded $*$ -algebra with the grading and the involution induced from A . It is clear that the direct product of graded algebras with involution is also a graded algebra with involution (the grading and the involution are component-wise).

We can also define the notion of a graded $*$ -identity for a G -graded algebra with a graded involution. The free associative non-unitary algebra $\mathfrak{F}^G = F\langle Y^G, Z^G \rangle$ generated by the set $Y^G \cup Z^G = \{y_{i\theta} | \theta \in G, i \in \mathbb{N}\} \cup \{z_{i\theta} | \theta \in G, i \in \mathbb{N}\}$ has the involution defined by (1) for monomials in $Y^G \cup Z^G$. We assume that $y_{j\theta}^* = y_{j\theta}$, and $z_{j\theta}^* = -z_{j\theta}$ (for all $\theta \in G, i \in \mathbb{N}$). The G -grading on \mathfrak{F}^G is defined naturally by the rule $\deg_G a_{i_1} a_{i_2} \cdots a_{i_n} = \deg_G a_{i_1} \cdots \deg_G a_{i_n}$, where $\deg_G y_{i\theta} = \deg_G z_{j\theta} = \theta$, $a_j \in Y^G \cup Z^G$. It is clear that the involution (1) is graded. The algebra \mathfrak{F}^G is the free associative G -graded algebra with graded involution. Its elements are called *graded $*$ -polynomials*. Variables $y_{i\theta} \in Y^G$, $z_{j\theta} \in Z^G$ are \widehat{G} -homogeneous. Their complete degrees are $\deg_{\widehat{G}} y_{i\theta} = (+, \theta)$, $\deg_{\widehat{G}} z_{i\theta} = (-, \theta)$, $\theta \in G$. Let us denote also $Y_\theta = \{y_{i\theta} | i \in \mathbb{N}\}$, and $Z_\theta = \{z_{i\theta} | i \in \mathbb{N}\}$ for any $\theta \in G$.

Let $f = f(x_1, \dots, x_n) \in F\langle Y^G, Z^G \rangle$ be a non-trivial graded $*$ -polynomial ($x_i \in Y^G \cup Z^G$). We say that a graded $*$ -algebra A satisfies the *graded $*$ -identity* (or *graded identity with involution*) $f = 0$ iff $f(a_1, \dots, a_n) = 0$ for all \widehat{G} -homogeneous elements $a_i \in A_{\theta_i}^{\delta_i}$ of the corresponding complete degrees $\deg_{\widehat{G}} a_i = \deg_{\widehat{G}} x_i = (\delta_i, \theta_i)$, $\delta_i \in \{+, -\}$, $\theta_i \in G$ ($i = 1, \dots, n$).

Denote by $\text{Id}^{gi}(A) \trianglelefteq F\langle Y^G, Z^G \rangle$ the ideal of all graded identities with involution of a graded $*$ -algebra A . It is clear that $\text{Id}^{gi}(A)$ is a two-side graded $*$ -ideal of $F\langle Y^G, Z^G \rangle$ closed under graded $*$ -endomorphisms of $F\langle Y^G, Z^G \rangle$. We call such ideals *giT-ideals* (see [35]). Conversely, any giT-ideal I of $F\langle Y^G, Z^G \rangle$ is the ideal of graded $*$ -identities of the graded algebra with involution $F\langle Y^G, Z^G \rangle / I$. Given a set $S \subseteq F\langle Y^G, Z^G \rangle$ of graded $*$ -polynomials denote by $giT[S]$ the giT-ideal generated by S . Similarly to case of non-graded $*$ -identities, we have that $\text{Id}^{gi}(A_1 \times \cdots \times A_\rho) = \bigcap_{i=1}^{\rho} \text{Id}^{gi}(A_i)$ for the direct product $A_1 \times \cdots \times A_\rho$ of graded $*$ -algebras.

Given a giT-ideal Γ consider the family \mathfrak{W}_Γ^G of all associative G -graded $*$ -algebras

that satisfy $f = 0$ for any $f \in \Gamma$. We call \mathfrak{V}_Γ^G a *graded $*$ -variety defined by Γ* . If $\Gamma = \text{Id}^{gi}(A)$ then we say that the graded $*$ -algebra A generates the graded $*$ -variety $\mathfrak{V}_\Gamma^G = \mathfrak{V}^G(A)$. Particularly, $\mathfrak{V}_\Gamma^G = \mathfrak{V}^G(F\langle Y^G, Z^G \rangle / \Gamma)$. Moreover, the algebra $\mathfrak{F}_\Gamma = F\langle Y^G, Z^G \rangle / \Gamma$ is the relatively free algebra of the graded $*$ -variety \mathfrak{V}_Γ^G . It is clear that $B \in \mathfrak{V}^G(A)$ for a graded $*$ -algebra B whenever $\text{Id}^{gi}(A) \subseteq \text{Id}^{gi}(B)$. Any graded $*$ -variety is closed under taking graded $*$ -subalgebras, graded $*$ -homomorphic images, and direct products.

Let $Y_\nu^G = \{y_{i\theta} | \theta \in G, 1 \leq i \leq \nu\}$, $Z_\nu^G = \{z_{i\theta} | \theta \in G, 1 \leq i \leq \nu\}$ be two finite sets, $\nu \in \mathbb{N}$. We also consider the free G -graded algebra with involution $F\langle Y_\nu^G, Z_\nu^G \rangle$ of rank ν generated by $Y_\nu^G \cup Z_\nu^G$ and the relatively free algebra of rank ν $\mathfrak{F}_{\nu, \Gamma} = F\langle Y_\nu^G, Z_\nu^G \rangle / (\Gamma \cap F\langle Y_\nu^G, Z_\nu^G \rangle)$ for the graded $*$ -variety \mathfrak{V}_Γ^G .

Observe that omitting indices by the elements of the group G in the structures of the free graded $*$ -algebra, graded $*$ -identities and graded $*$ -varieties we obtain the notions of non-graded identities with involution and non-graded $*$ -varieties. Notice that in both cases (graded and non-graded) variables of the set Y are reserved for symmetric elements, and variables Z for skew-symmetric. Two G -graded algebras with involution A and B are called *gi-equivalent*, $A \sim_{gi} B$, if $\text{Id}^{gi}(A) = \text{Id}^{gi}(B)$. Non-graded algebras with involution A and B are *$*$ PI-equivalent*, $A \sim_* B$, if $\text{Id}^*(A) = \text{Id}^*(B)$. Given a *gi*-T-ideal ($*$ T-ideal) Γ and graded (non-graded) $*$ -polynomials f, g we write $f = g \pmod{\Gamma}$ if $f - g \in \Gamma$.

If we have a graded $*$ -algebra A then we assume that $\text{Id}^*(A) \subseteq \text{Id}^{gi}(A)$. Namely, for a non-graded $*$ -polynomial $f(y_1, \dots, y_n, z_1, \dots, z_m) \in F\langle Y, Z \rangle$ we assume $f \in \text{Id}^{gi}(A)$ whenever $f(\sum_{\theta \in G} y_{1\theta}, \dots, \sum_{\theta \in G} y_{n\theta}, \sum_{\theta \in G} z_{1\theta}, \dots, \sum_{\theta \in G} z_{m\theta}) \in \text{Id}^{gi}(A)$. Particularly, for a multilinear non-graded $*$ -polynomial $f(y_1, \dots, y_n, z_1, \dots, z_m) \in F\langle Y, Z \rangle$ we have $f \in \text{Id}^{gi}(A)$ if and only if $f(y_{1\theta_1}, \dots, y_{n\theta_n}, z_{1\theta_{n+1}}, \dots, z_{m\theta_{n+m}}) \in \text{Id}^{gi}(A)$ for all $(\theta_1, \dots, \theta_{n+m}) \in G^{n+m}$. Thus if $A \sim_{gi} B$ for G -graded $*$ -algebras A, B then we have also that $A \sim_* B$.

Note that the set $X^G = \{x_{i\theta} = y_{i\theta} + z_{i\theta} | i \in \mathbb{N}, \theta \in G\}$ generates in \mathfrak{F}^G a G -graded subalgebra $F\langle X^G \rangle$ which is isomorphic to the free associative G -graded algebra ([33]). Thus the ideal $\text{Id}^G(A)$ of graded identities of A also lies in $\text{Id}^{gi}(A)$.

Recall that an algebra A is called *PI-algebra* if it satisfies a non-trivial ordinary polynomial identity (non-graded and without involution) (see [17], [18], [24], [27], [28]). It is clear that for a G -graded PI-algebra A with involution the T-ideal of ordinary polynomial identities $\text{Id}(A)$ also lies in $\text{Id}^{gi}(A)$. Moreover, we have that $\text{Id}(A) \subseteq \text{Id}^*(A) \subseteq \text{Id}^{gi}(A)$. Here for a polynomial $f(x_1, \dots, x_n) \in \text{Id}(A)$ we assume that $f \in \text{Id}^*(A)$ iff $f(y_1 + z_1, \dots, y_n + z_n) \in \text{Id}^*(A)$. This is the natural relation induced by the isomorphism (2) of $F\langle X^* \rangle$ and $F\langle Y, Z \rangle$ and inclusion $F\langle X \rangle \subseteq F\langle X^* \rangle$.

By Amitsur's theorem [2], [3] (see also [24]) any $*$ -algebra satisfying a non-trivial $*$ -identity is a PI-algebra. Thus any non-trivial $*$ T-ideal contains a non-trivial T-ideal. A G -graded $*$ -algebra can not be a PI-algebra in general (see for instance comments after Theorem 1 [33]). In general case a graded $*$ -algebra A is a PI-algebra iff the neutral component A_ϵ satisfies a non-trivial $*$ -identity, where ϵ is the unit element of G (it follows from [2], [3], and [4], [15]). This is equivalent to condition that A satisfies a non-trivial non-graded $*$ -identity.

The notion of degree of a graded or non-graded $*$ -polynomial is defined in the usual way. Using the multilinearization process as in the case of ordinary identities ([17], [18], [24]) we can show that any gi T-ideal or $*$ T-ideal over a field of characteristic zero is generated by multilinear polynomials (see also Lemma 1.1). Thus in our case it is enough to consider only multilinear identities.

The space of multilinear $*$ -polynomials of degree n has the form

$$P_n = \text{Span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n, x_i \in Y \cup Z\}.$$

Thus P_n is the direct sum of subspaces of multihomogeneous and multilinear polynomials depending on a fixed set of symmetric and skew-symmetric variables. When we consider $*$ -identities we can assume that a multilinear $*$ -identity depends on variables $\{y_1, \dots, y_k\}$, and $\{z_1, \dots, z_{n-k}\}$, $k = 0, \dots, n$. Denote by $P_{k,n-k}$ the subspace of all multilinear $*$ -polynomials $f(y_1, \dots, y_k, z_1, \dots, z_{n-k})$ for a fixed number k . Given a $*$ T-ideal $\Gamma \subseteq F\langle Y, Z \rangle$ the vector spaces $\Gamma_{k,n-k} = \Gamma \cap P_{k,n-k}$, and $P_{k,n-k}(\Gamma) = P_{k,n-k}/\Gamma_{k,n-k} \subseteq F\langle Y, Z \rangle/\Gamma$ has the natural structure of $(FS_k \otimes FS_{n-k})$ -modules. Here S_k and S_{n-k} act on symmetric and skew-symmetric variables independently renaming the variables (see, e.g., [20]).

Further we consider $(\mathbb{Z}/4\mathbb{Z})$ -graded algebras with involution and $(\mathbb{Z}/4\mathbb{Z})$ -graded $*$ -identities. We assume that $G = \mathbb{Z}/4\mathbb{Z}$, and use for it the additive notation. We also denote for brevity the group $\mathbb{Z}/4\mathbb{Z}$ by \mathbb{Z}_4 , and the free \mathbb{Z}_4 -graded $*$ -algebra $F\langle Y^{\mathbb{Z}_4}, Z^{\mathbb{Z}_4} \rangle$ by $\mathfrak{F}^{(4)}$.

Let us define the function $\eta : \mathbb{Z}_4 \rightarrow \{0, 1\}$ by the rules $\eta(\bar{0}) = \eta(\bar{1}) = 0$, $\eta(\bar{2}) = \eta(\bar{3}) = 1$. The next elementary properties of η can be checked directly

$$\begin{aligned} \eta(x) + \eta(y) &= \eta(x + y) + 1 \pmod{2} & \text{if } x, y \in \{\bar{1}, \bar{3}\}, \\ \eta(x) + \eta(y) &= \eta(x + y) \pmod{2} & \text{if } x \text{ or } y \text{ is even.} \end{aligned}$$

2 Grassmann \mathbb{Z}_4 -envelope of a graded $*$ -algebra.

Assume that $G = \mathbb{Z}_4$. Consider a \mathbb{Z}_4 -graded algebra $A = \bigoplus_{\theta \in \mathbb{Z}_4} A_\theta$.

Definition 2.1 *The algebra $E_4(A) = \bigoplus_{\theta \in \mathbb{Z}_4} A_\theta \otimes_F E_\theta$ is called Grassmann \mathbb{Z}_4 -envelope of A . Where $E = \bigoplus_{\theta \in \mathbb{Z}_4} E_\theta$ is the natural \mathbb{Z}_4 -grading of E defined in Example 1.2.*

The algebra $E_4(A)$ is also \mathbb{Z}_4 -graded with the grading $(E_4(A))_\theta = A_\theta \otimes_F E_\theta$, $\theta \in \mathbb{Z}_4$. If A has a graded involution $*_A$ then the F -linear involution $*$ on $E_4(A)$ is defined by the rules $(a \otimes g)^* = a^{*_A} \otimes g^{*_E}$, where $*_E$ is the canonic involution on E (see Example 1.3). Hence $(a_\theta \otimes g_\theta)^* = (-1)^{\eta(\theta)} a_\theta^{*_A} \otimes g_\theta$ for any $a_\theta \in A_\theta$, $g_\theta \in E_\theta$, $\theta \in \mathbb{Z}_4$. It is clear that $E_4(A)^\delta = \bigoplus_{\theta \in G} (E_4(A))_\theta^\delta$, $\delta \in \{+, -\}$, where

$$\begin{aligned} (E_4(A))_\theta^+ &= \text{Span}_F\{a_\theta \otimes g_\theta \mid a_\theta \in A_\theta^+, g_\theta \in E_\theta\} \text{ and} \\ (E_4(A))_\theta^- &= \text{Span}_F\{a_\theta \otimes g_\theta \mid a_\theta \in A_\theta^-, g_\theta \in E_\theta\} & \text{if } \theta \in \{\bar{0}, \bar{1}\}; \\ (E_4(A))_\xi^+ &= \text{Span}_F\{a_\xi \otimes g_\xi \mid a_\xi \in A_\xi^-, g_\xi \in E_\xi\} \text{ and} \\ (E_4(A))_\xi^- &= \text{Span}_F\{a_\xi \otimes g_\xi \mid a_\xi \in A_\xi^+, g_\xi \in E_\xi\} & \text{if } \xi \in \{\bar{2}, \bar{3}\}. \end{aligned} \tag{4}$$

Let us define some transformations of multilinear \mathbb{Z}_4 -graded $*$ -polynomials. Denote by $X^{\text{od}} = Y_{\bar{1}} \cup Z_{\bar{1}} \cup Y_{\bar{3}} \cup Z_{\bar{3}}$ the subset of all variables, odd in respect to the \mathbb{Z}_4 -grading, and by $X^{\text{ev}} = Y_{\bar{0}} \cup Z_{\bar{0}} \cup Y_{\bar{2}} \cup Z_{\bar{2}}$ the subset of all \mathbb{Z}_4 -even variables. Fix on X^{od} the linear order $y_{1\bar{1}} < y_{2\bar{1}} < \dots < z_{1\bar{1}} < z_{2\bar{1}} < \dots < y_{1\bar{3}} < y_{2\bar{3}} < \dots < z_{1\bar{3}} < z_{2\bar{3}} < \dots$. Assume that $f \in \mathfrak{F}^{(4)}$ is a multilinear graded $*$ -polynomial. Then f is uniquely represented in the form

$$f = \sum_u \sum_{\sigma \in S_k} \alpha_{\sigma, u} u_1 x_{\sigma(1)} u_2 x_{\sigma(2)} \cdots x_{\sigma(k)} u_{k+1}, \quad (5)$$

where $x_j \in X^{\text{od}}$, and $u = u_1 u_2 \cdots u_{k+1}$ is a multilinear monomial over X^{ev} , possibly empty, $k \geq 0$. Then we assume that

$$\mathfrak{s}(f) = \sum_u \sum_{\sigma \in S_k} (-1)^\sigma \alpha_{\sigma, u} u_1 x_{\sigma(1)} u_2 x_{\sigma(2)} \cdots x_{\sigma(k)} u_{k+1}. \quad (6)$$

Consider a collection of variables $(y_\theta, z_\theta) = (y_{1\theta}, \dots, y_{n_\theta\theta}, z_{1\theta}, \dots, z_{m_\theta\theta})$ of \mathbb{Z}_4 -degree θ . Then for a multilinear graded $*$ -polynomial $f = f(y_{\bar{0}}, z_{\bar{0}}, y_{\bar{1}}, z_{\bar{1}}, y_{\bar{2}}, z_{\bar{2}}, y_{\bar{3}}, z_{\bar{3}})$

$$\mathfrak{t}(f) = f \left| \begin{array}{l} y_{i\bar{2}} := z_{i\bar{2}}, y_{i\bar{3}} := z_{i\bar{3}}, \\ z_{i\bar{2}} := y_{i\bar{2}}, z_{i\bar{3}} := y_{i\bar{3}} \end{array} \right. = f(y_{\bar{0}}, z_{\bar{0}}, y_{\bar{1}}, z_{\bar{1}}, z_{\bar{2}}, y_{\bar{2}}, z_{\bar{3}}, y_{\bar{3}}) \quad (7)$$

is the respective exchange of the variables $y \in Y_\theta$ by $z \in Z_\theta$, and z by y of \mathbb{Z}_4 -degrees $\theta = \bar{2}$ and $\bar{3}$. Observe that $\mathfrak{t}(y_{1\theta}, \dots, y_{n_\theta\theta}, z_{1\theta}, \dots, z_{m_\theta\theta}) = (z_{1\theta}, \dots, z_{m_\theta\theta}, y_{1\theta}, \dots, y_{n_\theta\theta})$. It is clear that $\mathfrak{s}, \mathfrak{t}$ are linear operators on the space of multilinear $*$ - \mathbb{Z}_4 -polynomials. These operators satisfy the relations $\mathfrak{s}^2 = \mathfrak{t}^2 = \text{id}$, $\mathfrak{st} = \pm \mathfrak{ts}$, where id is the identical transformation, and the sign in the second formula is defined by the permutation of variables $y_{\bar{3}}, z_{\bar{3}}$ induced by applying of \mathfrak{t} . Then we denote

$$\widetilde{f} = \mathfrak{st}(f) \quad (8)$$

for a multilinear $*$ - \mathbb{Z}_4 -polynomial $f \in \mathfrak{F}^{(4)}$. It is clear that $\widetilde{\widetilde{f}} = \pm f$ for any multilinear $f \in \mathfrak{F}^{(4)}$. Moreover, we have the next Lemma.

Lemma 2.2 *A \mathbb{Z}_4 -graded algebra A with involution satisfies a multilinear \mathbb{Z}_4 -graded $*$ -identity $f = 0$ if and only if $E_4(A)$ satisfies $\widetilde{f} = 0$.*

Proof. Assume that f is a multilinear \mathbb{Z}_4 -graded $*$ -polynomial. Then

$$f = \sum_w \alpha_w w((y_{i_1\bar{0}}), (z_{i_2\bar{0}}), (y_{i_3\bar{1}}), (z_{i_4\bar{1}}), (y_{i_5\bar{2}}), (z_{i_6\bar{2}}), (y_{i_7\bar{3}}), (z_{i_8\bar{3}}))), \quad \alpha_w \in F,$$

where $w = w((y_{i_1\bar{0}}), (z_{i_2\bar{0}}), (y_{i_3\bar{1}}), (z_{i_4\bar{1}}), (y_{i_5\bar{2}}), (z_{i_6\bar{2}}), (y_{i_7\bar{3}}), (z_{i_8\bar{3}})))$ is a multilinear monomial, $y_\theta = (y_{i\theta})$, $z_\theta = (z_{i\theta})$, $\theta \in \mathbb{Z}_4$. Therefore,

$$\begin{aligned} \widetilde{f} &= \sum_w (-1)^{\sigma_{\widehat{w}}} \alpha_w \widehat{w}, \quad \text{where} \\ \widehat{w} &= \mathfrak{t}(w) = w((y_{i_1\bar{0}}), (z_{i_2\bar{0}}), (y_{i_3\bar{1}}), (z_{i_4\bar{1}}), (z_{i_5\bar{2}}), (\widetilde{y_{i_6\bar{2}}}), (z_{i_7\bar{3}}), (y_{i_8\bar{3}}))) = \\ &= u_{\widehat{w}1} x_{\sigma_{\widehat{w}}(1)} u_{\widehat{w}2} x_{\sigma_{\widehat{w}}(2)} \cdots x_{\sigma_{\widehat{w}}(k)} u_{\widehat{w}k+1}. \end{aligned}$$

The last formula gives the representation (5) of the monomial \widehat{w} ; here $\sigma_{\widehat{w}} \in S_k$, $x_j \in X^{\circ\circ}$, and $u_{\widehat{w}j}$ are monomials over $X^{\circ\circ}$, possibly empty. Since \widetilde{f} is multilinear then it is enough to consider its evaluations by elements $a \otimes g$, where $a \in A_\theta$, $g \in E_\theta$. Taking into account (4) we need to consider evaluations of the form

$$\begin{aligned} y_{i_1\bar{0}} &= b_{i_1\bar{0}} \otimes h_{i_1\bar{0}}, & z_{i_2\bar{0}} &= c_{i_2\bar{0}} \otimes \tilde{h}_{i_2\bar{0}}, \\ y_{i_3\bar{1}} &= b_{i_3\bar{1}} \otimes g_{i_3\bar{1}}, & z_{i_4\bar{1}} &= c_{i_4\bar{1}} \otimes \tilde{g}_{i_4\bar{1}}, \\ y_{i_6\bar{2}} &= c_{i_6\bar{2}} \otimes h_{i_6\bar{2}}, & z_{i_5\bar{2}} &= b_{i_5\bar{2}} \otimes \tilde{h}_{i_5\bar{2}}, \\ y_{i_8\bar{3}} &= c_{i_8\bar{3}} \otimes g_{i_8\bar{3}}, & z_{i_7\bar{3}} &= b_{i_7\bar{3}} \otimes \tilde{g}_{i_7\bar{3}}, \end{aligned} \quad (9)$$

where $b_{j\theta} \in A_\theta^+$, $c_{j\theta} \in A_\theta^-$, and elements $h_{j\theta}, g_{j\theta}, \tilde{h}_{j\theta}, \tilde{g}_{j\theta} \in E_\theta$ involve disjoint sets of generators of E . Assume that $(a_1 \otimes g_1, \dots, a_n \otimes g_n)$ is an evaluation of \widetilde{f} of the type (9) (for corresponding elements $a_i \in A$, $g_i \in E$). Observe that the elements $h_{i\theta}, \tilde{h}_{j\xi} \in E_{\bar{0}} \cup E_{\bar{2}}$ commute with any element of E , and the elements $g_{i\theta}, \tilde{g}_{j\xi} \in E_{\bar{1}} \cup E_{\bar{3}}$ anti-commute among themselves. Then we obtain

$$\begin{aligned} \widehat{w}(a_1 \otimes g_1, \dots, a_n \otimes g_n) &= w((b_{i_1\bar{0}} \otimes h_{i_1\bar{0}}), (c_{i_2\bar{0}} \otimes \tilde{h}_{i_2\bar{0}}), (b_{i_3\bar{1}} \otimes g_{i_3\bar{1}}), \\ & (c_{i_4\bar{1}} \otimes \tilde{g}_{i_4\bar{1}}), (b_{i_5\bar{2}} \otimes \tilde{h}_{i_5\bar{2}}), (c_{i_6\bar{2}} \otimes h_{i_6\bar{2}}), (b_{i_7\bar{3}} \otimes \tilde{g}_{i_7\bar{3}}), (c_{i_8\bar{3}} \otimes g_{i_8\bar{3}})) = \\ & w((b_{i_1\bar{0}}), (c_{i_2\bar{0}}), (b_{i_3\bar{1}}), (c_{i_4\bar{1}}), (b_{i_5\bar{2}}), (c_{i_6\bar{2}}), (b_{i_7\bar{3}}), (c_{i_8\bar{3}})) \otimes \widehat{w}(g_1, \dots, g_n) = \\ & w(\tilde{a}_1, \dots, \tilde{a}_n) \otimes u_{\widehat{w}1}(h, \tilde{h}) g'_{\sigma_{\widehat{w}}(1)} u_{\widehat{w}2}(h, \tilde{h}) g'_{\sigma_{\widehat{w}}(2)} \cdots g'_{\sigma_{\widehat{w}}(k)} u_{\widehat{w}k+1}(h, \tilde{h}) = \\ & (-1)^{\sigma_{\widehat{w}}} w(\tilde{a}_1, \dots, \tilde{a}_n) \otimes g_1 \cdots g_n. \end{aligned}$$

Where $(\tilde{a}_1, \dots, \tilde{a}_n) = ((b_{i_1\bar{0}}), (c_{i_2\bar{0}}), (b_{i_3\bar{1}}), (c_{i_4\bar{1}}), (b_{i_5\bar{2}}), (c_{i_6\bar{2}}), (b_{i_7\bar{3}}), (c_{i_8\bar{3}}))$ are arbitrary \widehat{G} -homogeneous elements of A , $u_{\widehat{w}j}(h, \tilde{h})$ are monomials $u_{\widehat{w}j}$ evaluated by elements $h_{i\theta}, \tilde{h}_{j\xi} \in E_{\bar{0}} \cup E_{\bar{2}}$, and the k -tuple $(g'_1, \dots, g'_k) = ((g_{i_3\bar{1}}), (\tilde{g}_{i_4\bar{1}}), (\tilde{g}_{i_7\bar{3}}), (g_{i_8\bar{3}}))$. Therefore,

$$\begin{aligned} \widetilde{f}(a_1 \otimes g_1, \dots, a_n \otimes g_n) &= \sum_w (-1)^{\sigma_{\widehat{w}}} \alpha_w \widehat{w}(a_1 \otimes g_1, \dots, a_n \otimes g_n) = \\ \sum_w (-1)^{\sigma_{\widehat{w}}} (-1)^{\sigma_{\widehat{w}}} \alpha_w w(\tilde{a}_1, \dots, \tilde{a}_n) \otimes g_1 \cdots g_n &= f(\tilde{a}_1, \dots, \tilde{a}_n) \otimes g_1 \cdots g_n. \end{aligned}$$

Thus $\widetilde{f}(a_1 \otimes g_1, \dots, a_n \otimes g_n) = 0$ for any evaluation (9) if and only if $f(\tilde{a}_1, \dots, \tilde{a}_n) = 0$ for all appropriate $\tilde{a}_i \in A_{\theta_i}^{\delta_i}$, $\delta_i \in \{+, -\}$, $\theta_i \in \mathbb{Z}_4$. \square

Definition 2.3 Given a giT -ideal $\Gamma \subseteq \mathfrak{F}^{(4)}$ denote by $\widetilde{\Gamma}$ the giT -ideal generated by the set $S = \{\widetilde{f} | f \in \Gamma \cap (\cup_{n \geq 1} P_n)\}$ of \mathfrak{st} -images of all multilinear polynomials from Γ .

Lemma 2.2 along with properties of the operators \mathfrak{s} , \mathfrak{t} immediately implies the following.

Lemma 2.4 Given a giT -ideal $\Gamma \subseteq \mathfrak{F}^{(4)}$ we have that $\Gamma = \text{Id}^{g^i}(A)$ for a \mathbb{Z}_4 -graded \ast -algebra A iff $\widetilde{\Gamma} = \text{Id}^{g^i}(E_4(A))$. Besides that, $\widetilde{\widetilde{\Gamma}} = \Gamma$.

Hence, we have that $A \sim_{gi} B$ for \mathbb{Z}_4 -graded $*$ -algebras A, B if and only if $E_4(A) \sim_{gi} E_4(B)$. And $E_4(E_4(A)) \sim_{gi} A$ for any \mathbb{Z}_4 -graded algebra A with involution. The last property is also a simple consequence of the facts that $E_4(E_4(A)) = \bigoplus_{\theta \in \mathbb{Z}_4} A_\theta \otimes_F E_\theta \otimes_F E_\theta$, and the algebra $E_4(E) = \bigoplus_{\theta \in \mathbb{Z}_4} E_\theta \otimes_F E_\theta$ is commutative and non-nilpotent.

Remark 2.5 Since $E_4(A)$ is a subalgebra of $A \otimes_F E$ then by Regev's theorem [31] we have that $E_4(A)$ is a PI-algebra if and only if A is a PI-algebra. Particularly, consider a $*$ -variety \mathfrak{V} . Assume that \mathfrak{V} is defined by a $*$ -T-ideal $\Gamma \subseteq F\langle Y, Z \rangle$, and $\Gamma = \text{Id}^*(A)$ for an algebra with involution A . Denote by $\mathfrak{V}^{\mathbb{Z}_4}$ the class of all associative \mathbb{Z}_4 -graded F -algebras B with involution such that $E_4(B) \in \mathfrak{V}$. It is clear from Lemma 2.4 that $\mathfrak{V}^{\mathbb{Z}_4}$ is a \mathbb{Z}_4 -graded $*$ -variety defined by the gi -T-ideal Γ_1 of \mathbb{Z}_4 -graded $*$ -identities of the \mathbb{Z}_4 -graded algebra with involution $A \otimes_F E = \bigoplus_{\theta \in \mathbb{Z}_4} A \otimes_F E_\theta$. The gi -T-ideal $\Gamma_1 = \tilde{\Gamma}_2$, where Γ_2 is the gi -T-ideal generated by Γ , i.e.

$$\Gamma_2 = \Gamma^{\mathbb{Z}_4} = giT[S_\Gamma] \quad \text{for} \quad S_\Gamma = \{ f |_{y_i := \sum_{\theta \in \mathbb{Z}_4} y_{i\theta}, z_i := \sum_{\theta \in \mathbb{Z}_4} z_{i\theta}, \forall i} \mid f \in \Gamma \}. \quad (10)$$

3 Alternating and symmetrizing polynomials.

Let $f = f(s_1, \dots, s_k, x_1, \dots, x_n) \in F\langle Y, Z \rangle$ be a multilinear polynomial. Assume that $S = \{s_1, \dots, s_k\} \subseteq Y$ or $S \subseteq Z$. We say that f is alternating in S , if $f(s_{\sigma(1)}, \dots, s_{\sigma(k)}, x_1, \dots, x_n) = (-1)^\sigma f(s_1, \dots, s_k, x_1, \dots, x_n)$ holds for any permutation $\sigma \in S_k$.

For any multilinear polynomial with involution $g(s_1, \dots, s_k, x_1, \dots, x_n)$ we construct a multilinear polynomial f alternating in $S = \{s_1, \dots, s_k\}$ by setting

$$f(s_1, \dots, s_k, x_1, \dots, x_n) = \mathcal{A}_S(g) = \sum_{\sigma \in S_k} (-1)^\sigma g(s_{\sigma(1)}, \dots, s_{\sigma(k)}, x_1, \dots, x_n).$$

The corresponding mapping \mathcal{A}_S is a linear transformation of multilinear $*$ -polynomials. We call it the alternator. Any $*$ -polynomial f alternating in S can be decomposed as $f = \sum_{i=1}^m \alpha_i \mathcal{A}_S(u_i)$, where the u_i 's are monomials, $\alpha_i \in F$.

We say that a multilinear $*$ -polynomial $f(s_1, \dots, s_k, x_1, \dots, x_n)$ is symmetrizing in the set $S = \{s_1, \dots, s_k\}$ ($S \subseteq Y$ or $S \subseteq Z$), if $f(s_{\sigma(1)}, \dots, s_{\sigma(k)}, x_1, \dots, x_n) = f(s_1, \dots, s_k, x_1, \dots, x_n)$ for any $\sigma \in S_k$.

For any multilinear $*$ -polynomial $g(s_1, \dots, s_k, x_1, \dots, x_n)$ the multilinear $*$ -polynomial

$$f(s_1, \dots, s_k, x_1, \dots, x_n) = \mathcal{E}_S(g) = \sum_{\sigma \in S_k} g(s_{\sigma(1)}, \dots, s_{\sigma(k)}, x_1, \dots, x_n).$$

is symmetrizing in S . \mathcal{E}_S is also a linear transformation of multilinear $*$ -polynomials. It is called the symmetrizer. Any multilinear $*$ -polynomial f symmetrizing in S can be written as $f = \sum_{i=1}^m \alpha_i \mathcal{E}_S(u_i)$, where the u_i 's are monomials of f , and $\alpha_i \in F$. Properties of alternating and symmetrizing polynomials with involution are similar to that of ordinary polynomials (see, e.g., [17], [24], [28]).

Particularly, a multilinear $*$ -polynomial $f(s_1, \dots, s_k, x_1, \dots, x_n)$ is symmetrizing in variables $S = \{s_1, \dots, s_k\}$ iff $f(s_1, \dots, s_k, x_1, \dots, x_n)$ is the full linearization in the variable s of the non-zero polynomial $\hat{f} = \frac{1}{k!} f(s, \dots, s, x_1, \dots, x_n)$. Moreover, $\hat{f} = \sum_{(v)} \alpha_{(v)} v_0 s v_1 s \cdots s v_k$, whenever $f = \sum_{(v)} \sum_{\sigma \in S_k} \alpha_{(v)} v_0 s_{\sigma(1)} v_1 s_{\sigma(2)} \cdots s_{\sigma(k)} v_k$, where monomials v_0, v_1, \dots, v_k (possibly empty) do not depend on S , $\alpha_{(v)} \in F$.

Similarly we can consider graded $*$ -polynomials alternating or symmetrizing in a set of variables $S \subseteq Y_\theta$ or $S \subseteq Z_\theta$ for any fixed $\theta \in G$ (see [35]).

Lemma 3.1 *Consider disjoint collections of variables $\bar{y} = \{y_1, \dots, y_n\} \subseteq Y_{\bar{0}} \cup Y_{\bar{1}}$, $\bar{z} = \{z_1, \dots, z_m\} \subseteq Z_{\bar{0}} \cup Z_{\bar{1}}$, and $\bar{t} = \{t_{11}, \dots, t_{1\hat{n}_1}, \dots, t_{\hat{k}1}, \dots, t_{\hat{k}\hat{n}_k}\}$, where $\{t_{i1}, \dots, t_{i\hat{n}_i}\} \subseteq Y_{\bar{1}}$, or $\{t_{i1}, \dots, t_{i\hat{n}_i}\} \subseteq Z_{\bar{1}}$ for any $i = 1, \dots, \hat{k}$. Let $f(\bar{y}, \bar{z}, \bar{t}) \in \mathfrak{F}^{(4)}$ be a multilinear graded $*$ -polynomial, which is alternating in any collection $\{t_{i1}, \dots, t_{i\hat{n}_i}\}$, $i = 1, \dots, \hat{k}$. Then the polynomial \tilde{f} depends on the same variables as f , and \tilde{f} is symmetrizing in $\{t_{i1}, \dots, t_{i\hat{n}_i}\}$ for any $i = 1, \dots, \hat{k}$.*

Proof. It is clear that $\tilde{f} = \mathfrak{s}(f)$. Also the polynomial f can be decomposed as

$$f = \sum_{\substack{(v), \\ \tau \in S_r}} \sum_{\substack{\sigma_i \in S_{\hat{n}_i}, \\ 1 \leq i \leq \hat{k}}} \alpha_{(v), \tau} (-1)^{\sigma_1} \cdots (-1)^{\sigma_{\hat{k}}} (\sigma_1 \cdots \sigma_{\hat{k}} \tau) v_0 x_1 v_1 x_2 \cdots x_r v_k,$$

where $x_j \in Y_{\bar{1}} \cup Z_{\bar{1}}$, v_j are monomials (possibly empty) over $Y_{\bar{0}} \cup Z_{\bar{0}}$, the permutation τ acts on the variables x_j , and the permutations σ_i acts on disjoint subsets of the set $\{x_1, \dots, x_r\}$ corresponding to the sets of variables $\{t_{i1}, \dots, t_{i\hat{n}_i}\}$, $\alpha_{(v), \tau} \in F$. Then

$$\begin{aligned} \tilde{f} &= \sum_{\substack{(v), \\ \tau \in S_r}} \sum_{\substack{\sigma_i \in S_{\hat{n}_i}, \\ 1 \leq i \leq \hat{k}}} \alpha_{(v), \tau} (-1)^{\sigma_1 \cdots \sigma_{\hat{k}}} (-1)^{\sigma_1 \cdots \sigma_{\hat{k}} \tau} (\sigma_1 \cdots \sigma_{\hat{k}} \tau) v_0 x_1 v_1 x_2 \cdots x_r v_k = \\ &= \sum_{\substack{(v), \\ \tau \in S_r}} (-1)^\tau \alpha_{(v), \tau} \sum_{\substack{\sigma_i \in S_{\hat{n}_i}, \\ 1 \leq i \leq \hat{k}}} (\sigma_1 \cdots \sigma_{\hat{k}}) v_0 x_{\tau(1)} v_1 x_{\tau(2)} \cdots x_{\tau(r)} v_k. \end{aligned}$$

Thus \tilde{f} is symmetrizing in any $\{t_{i1}, \dots, t_{i\hat{n}_i}\}$. \square

Given a $*$ T-ideal $\Gamma \trianglelefteq F\langle Y, Z \rangle$ the vector space $\Gamma_{n,m} = \Gamma \cap P_{n,m}$ of multilinear $*$ -polynomials $f(y_1, \dots, y_n, z_1, \dots, z_m) \in \Gamma$ has the structure of $(FS_n \otimes FS_m)$ -module defined by $(\sigma \otimes \tau)f(y_1, \dots, y_n, z_1, \dots, z_m) = f(y_{\sigma(1)}, \dots, y_{\sigma(n)}, z_{\tau(1)}, \dots, z_{\tau(m)})$ for any $(\sigma, \tau) \in S_n \times S_m$. The character of the quotient module $P_{n,m}(\Gamma) = P_{n,m}/\Gamma_{n,m} \subseteq F\langle Y, Z \rangle/\Gamma$ can be decomposed as $\chi_{n,m}(\Gamma) = \sum_{\lambda \vdash n, \mu \vdash m} \mathbf{m}_{\lambda, \mu} (\chi_\lambda \otimes \chi_\mu)$, where $\chi_\lambda \otimes \chi_\mu$ is the irreducible $S_n \times S_m$ -character associated to the pair (λ, μ) of partitions $\lambda \vdash n, \mu \vdash m$, $\mathbf{m}_{\lambda, \mu} \in \mathbb{Z}$ is a multiplicity (see for instance [20], [21], [24], [26]). An irreducible submodule of $P_{n,m}(\Gamma)$ corresponding to the pair (λ, μ) is generated by a non-zero polynomial $f_{\lambda, \mu} = (e_{T_\lambda} \otimes e_{T_\mu})f$, where $f \in P_{n,m}$, and $e_{T_\lambda} \in FS_n$, $e_{T_\mu} \in FS_m$ are the essential idempotents corresponding to the Young tableaux T_λ , and T_μ respectively (see Definition 2.2.12 [24]). We say that a multilinear $*$ -polynomial f corresponds to the pair of partitions (λ, μ) if $(FS_n \otimes FS_m) f = (FS_n \otimes FS_m) f_{\lambda, \mu}$. Particularly, the next observation holds.

Remark 3.2 Given a multilinear $*$ -polynomial $f \in P_{n,m}$ there exist a finite set of pairs (λ_j, μ_j) (not necessary different) of partitions $\lambda_j \vdash n$, $\mu_j \vdash m$ ($j = 1, \dots, k$) and multilinear $*$ -polynomials $g_{\lambda_j, \mu_j} \in P_{n,m}$ such that g_{λ_j, μ_j} corresponds to (λ_j, μ_j) , and the $*T$ -ideal generated by f can be decomposed as $*T[f] = \sum_{j=1}^k *T[g_{\lambda_j, \mu_j}]$.

Moreover, by Theorem 5.9 [22]

$$\chi_{n,m}(\Gamma) = \sum_{(\lambda, \mu) \in H_\Gamma} \mathbf{m}_{\lambda, \mu} (\chi_\lambda \otimes \chi_\mu), \quad (11)$$

where $H_\Gamma = (H(k_1, l_1), H(k_2, l_2))$ is a double hook corresponding to Γ . The hook $H(k, l)$ is the set of all partitions $\lambda = (\lambda_1, \dots, \lambda_s)$ satisfying the condition $\lambda_{k+1} \leq l$. Applying arguments of Lemma 2.5.6 [24] we always can assume that for any $(\lambda, \mu) \in H_\Gamma$ the set of variables of a polynomial $f_{\lambda, \mu}$ can be decomposed into disjoint unions $\{y_1, \dots, y_n\} = (\bigcup_{i=1}^{k'_1} Y'_i) \cup (\bigcup_{i=1}^{l'_1} T'_i)$, $\{z_1, \dots, z_m\} = (\bigcup_{i=1}^{k'_2} Z'_i) \cup (\bigcup_{i=1}^{l'_2} S'_i)$, where $k'_r \leq k_r$, $l'_r \leq l_r$ ($r = 1, 2$), and $f_{\lambda, \mu}$ is symmetrizing in any $Y'_i \subseteq Y$, $Z'_j \subseteq Z$ ($1 \leq i \leq k'_1$, $1 \leq j \leq k'_2$), and alternating in any $T'_i \subseteq Y$, $S'_j \subseteq Z$ ($1 \leq i \leq l'_1$, $1 \leq j \leq l'_2$). Notice that $\mathbf{m}_{\lambda, \mu} = 0$ in (11) means that $f_{\lambda, \mu} = (e_{T_\lambda} \otimes e_{T_\mu})f \in \Gamma$ for any Young tableaux T_λ, T_μ and for any $*$ -polynomial $f \in P_{n,m}$, (see, e.g., Theorem 2.4.5 [24]).

4 Classification theorems.

Theorem 4.1 Let F be a field of characteristic zero. Any proper $*T$ -ideal of the free associative F -algebra with involution is the ideal of identities with involution of the Grassmann \mathbb{Z}_4 -envelope of some finitely generated associative \mathbb{Z}_4 -graded PI-algebra with graded involution.

Proof. Let Γ be a proper $*T$ -ideal of $F\langle Y, Z \rangle$, and \mathfrak{V}_Γ the $*$ -variety defined by Γ . Consider the \mathbb{Z}_4 -graded $*$ -variety $\tilde{\mathfrak{V}}_\Gamma^{\mathbb{Z}_4}$ of all associative \mathbb{Z}_4 -graded $*$ -algebras B such that $E_4(B) \in \mathfrak{V}_\Gamma$. Assume that $H_\Gamma = (H(k_1, l_1), H(k_2, l_2))$ is the double hook corresponding to Γ [22]. Take $\nu = \max\{k_1, l_1, k_2, l_2\}$, and the relatively free algebra \mathcal{R} of the rank ν of the \mathbb{Z}_4 -graded $*$ -variety $\tilde{\mathfrak{V}}_\Gamma^{\mathbb{Z}_4}$. Then as in Remark 2.5 we have $\mathcal{R} = F\langle Y_\nu^{\mathbb{Z}_4}, Z_\nu^{\mathbb{Z}_4} \rangle / (\tilde{\Gamma}_2 \cap F\langle Y_\nu^{\mathbb{Z}_4}, Z_\nu^{\mathbb{Z}_4} \rangle)$, where $\Gamma_2 = \Gamma^{\mathbb{Z}_4} = giT[S_\Gamma]$ is defined by (10). By Remark 2.5 and Amitsur's theorem [2], [3] \mathcal{R} is a PI-algebra. Let us prove that $E_4(\mathcal{R})$ generates \mathfrak{V}_Γ .

It is clear that $\text{Id}^*(E_4(\mathcal{R})) \supseteq \Gamma$. Take a multilinear polynomial with involution $f(y_1, \dots, y_n, z_1, \dots, z_m) \in \text{Id}^*(E_4(\mathcal{R})) \cap P_{n,m}$. By Remark 3.2 we can assume that f corresponds to a pair of partitions (λ, μ) , where $\lambda \vdash n$, and $\mu \vdash m$. If $(\lambda, \mu) \notin H_\Gamma$ then $f \in \Gamma$ by Theorem 5.9 [22]. Suppose that $(\lambda, \mu) \in H_\Gamma$. Then similarly to Lemma 2.5.6 [24] we can assume that the set $\{y_1, \dots, y_n\} \in Y$ of the variables of f is divided on at most ν sets of symmetrized variables $\{y_{i_1 1}, \dots, y_{i_1 n_{i_1}}\}$ ($i_1 = 1, \dots, \nu$), and at most ν sets of alternated variables $\{t_{i_2 1}, \dots, t_{i_2 n_{i_2}}\}$ ($i_2 = 1, \dots, \nu$). Similarly, the set $\{z_1, \dots, z_m\}$ consists of at most ν sets of symmetrized variables $\{z_{j_1 1}, \dots, z_{j_1 m_{j_1}}\}$

$(j_1 = 1, \dots, \nu)$, and at most ν sets of alternated variables $\{s_{j_2 1}, \dots, s_{j_2 \hat{m}_{j_2}}\}$ ($j_2 = 1, \dots, \nu$). Thus, $f = f(\vec{y}, \vec{t}, \vec{z}, \vec{s})$, where

$$\begin{aligned}\vec{y} &= (y_{11}, \dots, y_{1n_1}, \dots, y_{\nu 1}, \dots, y_{\nu n_\nu}) \subseteq Y, \\ \vec{t} &= (t_{11}, \dots, t_{1\hat{n}_1}, \dots, t_{\nu 1}, \dots, t_{\nu \hat{n}_\nu}) \subseteq Y, \\ \vec{z} &= (z_{11}, \dots, z_{1m_1}, \dots, z_{\nu 1}, \dots, z_{\nu m_\nu}) \subseteq Z, \\ \vec{s} &= (s_{11}, \dots, s_{1\hat{m}_1}, \dots, s_{\nu 1}, \dots, s_{\nu \hat{m}_\nu}) \subseteq Z\end{aligned}\tag{12}$$

are disjoint collections of variables, and f is symmetrizing in any $\{y_{i1}, \dots, y_{in_i}\}$, and $\{z_{i1}, \dots, z_{im_i}\}$, and alternating in any $\{t_{i1}, \dots, t_{i\hat{n}_i}\}$, and $\{s_{i1}, \dots, s_{i\hat{m}_i}\}$ ($i = 1, \dots, \nu$).

Since $f \in \text{Id}^*(E_4(\mathcal{R}))$ then f is equal to zero in $E_4(\mathcal{R})$ for

$$\begin{aligned}y_{ij_1} &= \bar{y}_{i\bar{0}} \otimes h_{n \cdot i + j_1 \bar{0}}, & t_{ij_2} &= \bar{y}_{i\bar{1}} \otimes g_{n \cdot i + j_2 \bar{1}}, \\ z_{ij_3} &= \bar{z}_{i\bar{0}} \otimes \tilde{h}_{m \cdot i + j_3 \bar{0}}, & s_{ij_4} &= \bar{z}_{i\bar{1}} \otimes \tilde{g}_{m \cdot i + j_4 \bar{1}}, \\ i &= 1, \dots, \nu, & 1 \leq j_1 \leq n_i, & 1 \leq j_2 \leq \hat{n}_i, & 1 \leq j_3 \leq m_i, & 1 \leq j_4 \leq \hat{m}_i,\end{aligned}\tag{13}$$

where $\bar{y}_{i\theta} = y_{i\theta} + \mathcal{I}$, $\bar{z}_{i\theta} = z_{i\theta} + \mathcal{I}$, $y_{i\theta} \in Y_\theta$, and $z_{i\theta} \in Z_\theta$ are graded variables from $Y_\nu^{\mathbb{Z}_4} \cup Z_\nu^{\mathbb{Z}_4}$ of \mathbb{Z}_4 -degree $\theta \in \{\bar{0}, \bar{1}\}$, $\mathcal{I} = \tilde{\Gamma}_2 \cap F\langle Y_\nu^{\mathbb{Z}_4}, Z_\nu^{\mathbb{Z}_4} \rangle$, $h_{i\bar{0}}, \tilde{h}_{i\bar{0}} \in E_{\bar{0}}$, $g_{i\bar{1}}, \tilde{g}_{i\bar{1}} \in E_{\bar{1}}$ are elements of the Grassmann algebra depending on disjoint sets of generators. Let us denote $a^{(k)} = \underbrace{a, \dots, a}_k$ for any element a . Therefore, we obtain in

the algebra $E_4(\mathcal{R})$ the equalities

$$\begin{aligned}f|_{(13)} &= \bar{f}_3 \otimes g = 0, & \text{where} \\ \bar{f}_3 &= f_2(\bar{y}_{1\bar{0}}^{(n_1)}, \dots, \bar{y}_{\nu\bar{0}}^{(n_\nu)}, \bar{y}_{1\bar{1}}^{(\hat{n}_1)}, \dots, \bar{y}_{\nu\bar{1}}^{(\hat{n}_\nu)}, \bar{z}_{1\bar{0}}^{(m_1)}, \dots, \bar{z}_{\nu\bar{0}}^{(m_\nu)}, \bar{z}_{1\bar{1}}^{(\hat{m}_1)}, \dots, \bar{z}_{\nu\bar{1}}^{(\hat{m}_\nu)}).\end{aligned}\tag{14}$$

Here $f_2 = \tilde{f}_1$. Where the graded multilinear polynomial $f_1 = f(\vec{y}_{\bar{0}}, \vec{y}_{\bar{1}}, \vec{z}_{\bar{0}}, \vec{z}_{\bar{1}})$, with

$$\begin{aligned}\vec{y}_{\bar{0}} &= (y_{(1,1)\bar{0}}, \dots, y_{(1,n_1)\bar{0}}, \dots, y_{(\nu,1)\bar{0}}, \dots, y_{(\nu,n_\nu)\bar{0}}) \subseteq Y_{\bar{0}}, \\ \vec{y}_{\bar{1}} &= (y_{(1,1)\bar{1}}, \dots, y_{(1,\hat{n}_1)\bar{1}}, \dots, y_{(\nu,1)\bar{1}}, \dots, y_{(\nu,\hat{n}_\nu)\bar{1}}) \subseteq Y_{\bar{1}}, \\ \vec{z}_{\bar{0}} &= (z_{(1,1)\bar{0}}, \dots, z_{(1,m_1)\bar{0}}, \dots, z_{(\nu,1)\bar{0}}, \dots, z_{(\nu,m_\nu)\bar{0}}) \subseteq Z_{\bar{0}}, \\ \vec{z}_{\bar{1}} &= (z_{(1,1)\bar{1}}, \dots, z_{(1,\hat{m}_1)\bar{1}}, \dots, z_{(\nu,1)\bar{1}}, \dots, z_{(\nu,\hat{m}_\nu)\bar{1}}) \subseteq Z_{\bar{1}},\end{aligned}\tag{15}$$

is the result of the evaluation of the variables y_{ij} , z_{ij} of the polynomial f by the corresponding graded variables of the degree $\bar{0}$, and of the variables t_{ij} , s_{ij} by the graded variables $y_{(i,j)\bar{1}}$, $z_{(i,j)\bar{1}}$ of the degree $\bar{1}$ respectively. The element g in (14) is the product of all elements $h_{i\bar{0}}, \tilde{h}_{i\bar{0}}, g_{i\bar{1}}, \tilde{g}_{i\bar{1}}$ of the Grassmann algebra from (13).

Observe that by Lemma 3.1 the polynomial $f_2 = \tilde{f}_1$ is symmetrizing in any set of variables $y_{(i,1)\theta}, \dots, y_{(i,n'_i)\theta}$, and $z_{(i,1)\theta}, \dots, z_{(i,m'_i)\theta}$, for all $i = 1, \dots, \nu$, $\theta \in \{\bar{0}, \bar{1}\}$ (if $\theta = \bar{0}$ then $n'_i = n_i$, $m'_i = m_i$, otherwise $n'_i = \hat{n}_i$, $m'_i = \hat{m}_i$). The equality (14) means that the graded $*$ -polynomial

$$f_3 = f_2(y_{1\bar{0}}^{(n_1)}, \dots, y_{\nu\bar{0}}^{(n_\nu)}, y_{1\bar{1}}^{(\hat{n}_1)}, \dots, y_{\nu\bar{1}}^{(\hat{n}_\nu)}, z_{1\bar{0}}^{(m_1)}, \dots, z_{\nu\bar{0}}^{(m_\nu)}, z_{1\bar{1}}^{(\hat{m}_1)}, \dots, z_{\nu\bar{1}}^{(\hat{m}_\nu)})$$

belongs to $\tilde{\Gamma}_2 \cap F\langle Y_\nu^{\mathbb{Z}_4}, Z_\nu^{\mathbb{Z}_4} \rangle$. Thus, $f_3 \in \tilde{\Gamma}_2$. The polynomial $f_2 = \tilde{f}_1(\vec{y}_0, \vec{y}_1, \vec{z}_0, \vec{z}_1)$ is the full linearization of $\frac{1}{\alpha} \cdot f_3$, where $\alpha \in F$ is some nonzero coefficient which appears as the result of identifying of symmetrized variables. The variables of f_2 as in (15). Hence $f_2 \in \tilde{\Gamma}_2$.

Take the relatively free $*$ -algebra $\mathcal{L} = F\langle Y, Z \rangle / \Gamma$ of the $*$ -variety \mathfrak{V}_Γ , and consider the \mathbb{Z}_4 -graded $*$ -algebra $\mathcal{L} \otimes E = \bigoplus_{\theta \in \mathbb{Z}_4} \mathcal{L} \otimes E_\theta$. By Remark 2.5 $\mathcal{L} \otimes E$ satisfies the graded $*$ -identity $f_2(\vec{y}_0, \vec{y}_1, \vec{z}_0, \vec{z}_1) = 0$. Particularly, the evaluation

$$\begin{aligned} y_{(i,j_1)\bar{0}} &= \bar{y}_{ij_1} \otimes h_{n \cdot i + j_1} \bar{0}, & y_{(i,j_2)\bar{1}} &= \bar{t}_{ij_2} \otimes g_{n \cdot i + j_2} \bar{1}, \\ z_{(i,j_3)\bar{0}} &= \bar{z}_{ij_3} \otimes \tilde{h}_{m \cdot i + j_3} \bar{0}, & z_{(i,j_4)\bar{1}} &= \bar{s}_{ij_4} \otimes \tilde{g}_{m \cdot i + j_4} \bar{1}, \\ i &= 1, \dots, \nu, \quad 1 \leq j_1 \leq n_i, \quad 1 \leq j_2 \leq \hat{n}_i, \quad 1 \leq j_3 \leq m_i, \quad 1 \leq j_4 \leq \hat{m}_i \end{aligned} \quad (16)$$

gives the result $f_2|_{(16)} = \tilde{f}_1(\vec{y}, \vec{t}, \vec{z}, \vec{s}) \otimes g' = f(\vec{y}, \vec{t}, \vec{z}, \vec{s}) \otimes g' = 0$. Here $\vec{y}, \vec{t}, \vec{z}, \vec{s}$ is the sequence formed as in (12) by the elements $\bar{y}_{ij_1} = y_{ij_1} + \Gamma$, $\bar{t}_{ij_2} = t_{ij_2} + \Gamma$, $\bar{z}_{ij_3} = z_{ij_3} + \Gamma$, $\bar{s}_{ij_4} = s_{ij_4} + \Gamma$ ($i = 1, \dots, \nu$, $1 \leq j_1 \leq n_i$, $1 \leq j_2 \leq \hat{n}_i$, $1 \leq j_3 \leq m_i$, $1 \leq j_4 \leq \hat{m}_i$), where the variables $y_{ij_1}, t_{ij_2}, z_{ij_3}, s_{ij_4}$ are the same as in (12). The element g' is the product of all elements of the Grassmann algebra from (16) depending on disjoint sets of generators. Therefore, $f(\vec{y}, \vec{t}, \vec{z}, \vec{s}) = 0$ in \mathcal{L} , and $f \in \Gamma$. Hence $\text{Id}^*(E_4(\mathcal{R})) = \Gamma$. \square

We can reinforce the result similarly to the classical case of Kemer's theorems for PI-algebras [28] using Theorem 6.2 [35].

Theorem 4.2 *Let F be a field of characteristic zero. Any proper $*T$ -ideal of the free associative F -algebra with involution is the ideal of identities with involution of the Grassmann \mathbb{Z}_4 -envelope of some associative \mathbb{Z}_4 -graded algebra with graded involution, finite dimensional over F .*

Proof. If Γ is a proper $*T$ -ideal of $F\langle Y, Z \rangle$ then by Theorem 4.1 we have $\Gamma = \text{Id}^*(E_4(B))$ for some associative finitely generated \mathbb{Z}_4 -graded PI-algebra B with graded involution. Theorem 6.2 [35] states that there exists a finite dimensional over F \mathbb{Z}_4 -graded algebra C with graded involution which has the same graded $*$ -identities as B . Hence $E_4(B) \sim_{gi} E_4(C)$. Particularly, $\text{Id}^*(E_4(B)) = \text{Id}^*(E_4(C)) = \Gamma$. \square

For a finitely generated associative PI-algebra with involution we also have the next theorem.

Theorem 4.3 (Theorem 1 [34]) *Let F be a field of characteristic zero. Then a non-zero $*T$ -ideal of $*$ -identities of a finitely generated associative F -algebra with involution coincides with the $*T$ -ideal of $*$ -identities of some finite dimensional associative F -algebra with involution.*

Observe that Theorem 4.3 can be considered as a partial case of Theorem 4.2 assuming that the \mathbb{Z}_4 -grading is trivial.

5 Specht problem.

Theorem 4.2 yields that any associative algebra with involution over a field of characteristic zero has a finite base of $*$ -identities.

Theorem 5.1 *Let F be a field of characteristic zero. Any $*T$ -ideal of the free associative F -algebra with involution $F\langle Y, Z \rangle$ is finitely generated as a $*T$ -ideal.*

Proof. It is clear that $F\langle Y, Z \rangle$ is generated as a $*T$ -ideal by the set $\{y_1, z_1\}$, and the zero ideal is generated by the zero polynomial. Hence it is enough to prove the theorem for proper $*T$ -ideals.

Suppose that there exists a proper $*T$ -ideal $\Gamma \subseteq F\langle Y, Z \rangle$ which can not be finitely generated as a $*T$ -ideal. Then there exists an infinite sequence of multilinear $*$ -polynomials $\{f_i(x_1, \dots, x_{n_i})\}_{i \in \mathbb{N}} \subseteq \Gamma$, such that $\deg f_i < \deg f_j$ for any $i < j$, and $f_i \notin *T[f_1, \dots, f_{i-1}]$ for any $i \in \mathbb{N}$, where $x_j \in Y \cup Z$.

Given $i \in \mathbb{N}$ let us take the $*T$ -ideal $\Gamma_i \subseteq F\langle Y, Z \rangle$ generated by all consequences of the polynomial f_i of degrees strictly greater than $n_i = \deg f_i$. Consider the $*T$ -ideal $\tilde{\Gamma} = \sum_{i \in \mathbb{N}} \Gamma_i$. It is clear that for any $i \in \mathbb{N}$ we have that $f_i \notin \tilde{\Gamma}$. By Theorem 4.2 $\tilde{\Gamma}$ is the ideal of identities with involution of the Grassmann \mathbb{Z}_4 -envelope $E_4(C)$ of some finite dimensional over F \mathbb{Z}_4 -graded algebra C with graded involution.

By Lemma 3.1 [35] $C = B \oplus J$, where B is a \mathbb{Z}_4 -graded semisimple algebra with a graded involution, and $J = J(C)$ is a \mathbb{Z}_4 -graded nilpotent ideal of C . By [5], [36] B has the unit $1_B \in B_{\bar{0}}$, and 1_B is symmetric in respect to involution. Therefore, $E_4(C) = E_4(B) \oplus E_4(J)$, where $E_4(B)$ is a $*$ -subalgebra of $E_4(C)$, and $E_4(J)$ is a nilpotent $*$ -ideal of $E_4(C)$ of degree \mathcal{N} .

Let us take a polynomial $f_k(x_1, \dots, x_{n_k})$ of degree $n_k = \deg f_k > \mathcal{N}$. Consider any evaluation of the polynomial f_k of the type $x_i = a_i = c_{\theta_i} \otimes g_{\theta_i}$, $a_i \in E_4(C)^+$ if $x_i \in Y$, and $a_i \in E_4(C)^-$ if $x_i \in Z$. Where $c_{\theta_i} \in (B_{\theta_i}^+ \cup B_{\theta_i}^-) \cup (J_{\theta_i}^+ \cup J_{\theta_i}^-)$, $g_{\theta_i} \in E_{\theta_i}$, $\theta_i \in \mathbb{Z}_4$ for any $i = 1, \dots, n_k$.

If the element $c_{\theta_i} \in J_{\theta_i}^+ \cup J_{\theta_i}^-$ is radical for any $i = 1, \dots, n_k$ then $f_k(a_1, \dots, a_{n_k}) = 0$ in $E_4(C)$, since $n_k > \mathcal{N}$. Suppose that at least one of the variables of f_k is evaluated by an element of the type $b_{\theta} \otimes g_{\theta}$, where $b_{\theta} \in B_{\theta}$ is a semisimple element of C . Assume that $x_{\hat{r}} = a_{\hat{r}} = b_{\theta_{\hat{r}}} \otimes g_{\theta_{\hat{r}}}$ for any $b_{\theta_{\hat{r}}} \in B_{\theta_{\hat{r}}}$, $g_{\theta_{\hat{r}}} \in E_{\theta_{\hat{r}}}$, admitting $a_{\hat{r}} \in E_4(C)^+$ for $x_{\hat{r}} \in Y$ and $a_{\hat{r}} \in E_4(C)^-$ for $x_{\hat{r}} \in Z$.

The algebra $E_4(C)$ has the natural structure of $E_{\bar{0}}$ -module defined by $(c_{\theta} \otimes g_{\theta})g = c_{\theta} \otimes (g_{\theta}g)$, $g \in E_{\bar{0}}$, $c_{\theta} \in C_{\theta}$, $g_{\theta} \in E_{\theta}$, $\theta \in \mathbb{Z}_4$. This structure preserves the \mathbb{Z}_4 -grading and the involution $(\deg_{\mathbb{Z}_4} c_{\theta} \otimes (g_{\theta}g) = \deg_{\mathbb{Z}_4} c_{\theta} \otimes g_{\theta} = \theta$, $((c_{\theta} \otimes g_{\theta})g)^* = (c_{\theta} \otimes g_{\theta})^*g$. Moreover, $E_{\bar{0}}$ is the center of E . Hence for an element $\tilde{g}_0 \in E_0$ we obtain that

$$\begin{aligned} f_k(a_1, \dots, a_{\hat{r}}, \dots, a_{n_k})(2\tilde{g}_0) &= f_k(a_1, \dots, b_{\theta_{\hat{r}}} \otimes (g_{\theta_{\hat{r}}} \cdot 2\tilde{g}_0), \dots, a_{n_k}) = \\ f_k(a_1, \dots, (b_{\theta_{\hat{r}}} \otimes g_{\theta_{\hat{r}}}) \circ (1_B \otimes \tilde{g}_0), \dots, a_{n_k}) &= \tilde{f}_k(a_1, \dots, a_{n_k}, 1_B \otimes \tilde{g}_0) = 0. \end{aligned} \quad (17)$$

Here $\tilde{f}_k(x_1, \dots, x_{n_k}, y_0) = f_k(x_1, \dots, x_{\hat{r}} \circ y_0, \dots, x_{n_k}) \in \tilde{\Gamma} = \text{Id}^*(E_4(C))$, $y_0 \in Y_{\bar{0}}$. The equality $f_k(a_1, \dots, a_{n_k}) = 0$ directly follows from (17).

Since f_k is multilinear then it implies that $f_k \in \text{Id}^*(E_4(C)) = \widetilde{\Gamma}$. This contradicts to the construction of $\widetilde{\Gamma}$. Therefore, Γ is finitely generated as a $*$ T-ideal. \square

Observe that the usual Grassmann envelope of the superalgebras with superinvolution also can be considered in the context of the Specht problem and Classification theorems for identities with involution. We assume that results similar to Theorem 6.2 [35], and Theorems 4.1, 4.2 can be obtained also in this case.

Conjecture 5.1 *Let F be a field of characteristic zero, and $A = A_{\bar{0}} \oplus A_{\bar{1}}$ a finitely generated associative PI-superalgebra over F with superinvolution. Then there exists a finite dimensional over F associative superalgebra $C = C_{\bar{0}} \oplus C_{\bar{1}}$ with superinvolution which satisfies the same identities with superinvolution as A .*

Conjecture 5.2 *Let F be a field of characteristic zero. Then any associative F -algebra with involution satisfies the same $*$ -identities as the Grassmann envelope $E(C) = C_{\bar{0}} \otimes E_{\bar{0}} \oplus C_{\bar{1}} \otimes E_{\bar{1}}$ of some associative superalgebra $C = C_{\bar{0}} \oplus C_{\bar{1}}$ with superinvolution, finite dimensional over F .*

The confirmation of these conjectures could imply another solution of the Specht problem for $*$ -identities.

Author is deeply thankful to Ivan Shestakov and Antonio Giambruno for useful discussions and inspiration, and grateful to FAPESP for the financial support in this work.

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